TWO-DIMENSIONAL SPACES IN WHICH THERE EXIST CONTIGUOUS POINTS*

BY

E. C. KLIPPLE

In a recent number of The Rice Institute Pamphlet† R. L. Moore has formulated a set of axioms in terms of the undefined notions "point," "region," and "contiguous to." These axioms‡ (Axioms A, B, C, 0, 1, and 2 of this paper) serve as the basis for the proofs of a considerable number of theorems of ordinary point set theory, including a large proportion of the theorems of the first two chapters of Moore's book.§ Nevertheless, there exist spaces satisfying these axioms in which an arc may contain only a finite number of points and in which a region may consist of a finite number of points.

In the present paper a study is made of spaces which satisfy the above mentioned axioms and some additional axioms which restrict the spaces to being, in a certain sense, two dimensional. The ordinary euclidean plane is a space which satisfies all the axioms.

I wish to acknowledge my indebtedness to Professor R. L. Moore, and to thank him for suggesting the problem and for many helpful criticisms in the course of its development.

For definitions of terms used but not defined here the reader may refer to S.C.P.

DEFINITIONS. *A simple closed curve is a compact continuum, containing at least two distinct non-contiguous points, which is disconnected by the omission of any two of its non-contiguous points. A triune is a set of three distinct points such that each of them is contiguous to each of the others.*

AXIOM A. *No point is contiguous to itself.*

AXIOM B. *If the point A is contiguous to the point B, then B is contiguous to A.*

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† Vol. 23 (1936), no. 1. In the present treatment the abbreviation S.C.P. will be used to designate part 1 of this paper.

‡ It being understood that in the statement of Axiom 2 of S.C.P. the word "non-degenerate" is to be omitted. It is clear from the context that the retention of this word was not intended.

§ *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. In the present treatment the abbreviation P.S.T. will be used to designate this book.

AXIOM C. *If M is a closed point set and every point of the set H is contiguous to some point of M, then no point of $S-M$ is a limit point of H.*

AXIOM 0. *Every region is a point set.*

AXIOM 1. *There exists a sequence G_1, G_2, G_3, $\cdots$ such that* (1) *for each n, G_n is a collection of regions covering S,* (2) *for each n, G_{n+1} is a subcollection of G_n,* (3) *if R is any region whatsoever, and if X is a point of R and Y is a point of R either identical with X or not, then there exists a natural number m such that if g is any region belonging to the collection G_m and containing X, then $\bar{g}$ is a subset of $(R-Y)+X$,* (4) *if M_1, M_2, M_3, $\cdots$ is a sequence of closed point sets such that M_n contains M_{n+1} for each n and there exists a region g_n of the collection G_n such that M_n is a subset of $\bar{g}_n$ for each n, then there is at least one point common to all the point sets of the sequence M_1, M_2, M_3, $\cdots$.*

AXIOM 2. *If P is a point of a region R, there exists a connected domain containing P and lying in R.*

AXIOM 3. *If J is a simple closed curve or triune, then $S-J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them.*

In this paper I deal very frequently with complementary domains of simple closed curves and triunes. If J is a simple closed curve or triune and ω is a point of $S-J$, then by *the interior of J with respect to ω*, is meant that complementary domain of J which does not contain ω. Similarly, that complementary domain of J which contains ω is called *the exterior of J with respect to ω*. In case no ambiguity arises, the terms *interior* and *exterior* of J will be used without making specific reference to a point ω.

THEOREM 1. *Let J_1 and J_2 denote two point sets each of which is either a simple closed curve or a triune. Let I_1 and I_2 denote the interiors of J_1 and J_2, respectively, and suppose J_2 is a subset of J_1+I_1. Then I_2 is a subset of I_1.*

THEOREM 2. *If J is a simple closed curve, and AB separate C and D on J, and AXB is an arc such that the segment AXB is a subset of I, the interior of J, and no point of the segment AXB is constiguous to any point of $J-(A+B)$, then* (1) *I_1, the interior of $AXBCA$, is a subset of I,* (2) *the segment ADB is a subset of the exterior of $AXBCA$, and* (3) *I_1 has no point in common with I_2, the interior of $AXBDA$.*

THEOREM 3. *Under the hypothesis of Theorem 2, $I=I_1+I_2+$segment AXB.*

Proof. Suppose $I(I_1+I_2+\text{segment } AXB)=M$, where M is a non-vacuous point set. Let P be a point of M. By Theorem 38 of S.C.P. there exists an arc PX from P to X lying in I. The arc PX contains an arc PX' such that

$PX' - X'$ is a subset of M, and X' is a point of the segment AXB. Similarly, there exists an arc $P\omega$ lying in E_2, the exterior of $AXBDA$. The arc $P\omega$ contains an arc PC', such that $PC' - C'$ is a subset of M, and C' is a point of the segment ACB. Let T denote the first point in the order from X' to P which $X'P$ has in common with PC'. If no point of the interval $X'T$ of $X'P$, except possibly T, is contiguous to any point of $(TC' - T)$, where TC' denotes the interval of PC' from T to C', then $X'T + TC'$ is an arc from X' to C'. If there exist points of $X'T - T$ which are contiguous to points of $TC' - T$, there must be a first such point in the order from X' to T. For otherwise, there would be infinitely many such points, and the set of all such points would have a limit point in $X'T - T$. But by Axiom C, every limit point of such a set must belong to TC', and a contradiction is reached. Let W denote the first point of $X'T - T$ which is contiguous to a point of $TC' - T$. By Axiom C, W is contiguous to only a finite number of points of $TC' - T$. Let V denote the last point in the order from T to C' which is contiguous to W. Let $X'W$ denote the interval of $X'T$ from X' to W or the point X' according as W is not or is identical with X'. Let VC' denote the interval of TC' from V to C' or the point C' according as V is not or is identical with C'. The point set $X'W + VC'$ is an arc containing at least three points. Thus, in any case, the point set $PX' + PC'$ contains an arc $X'P'C'$, such that the segment $X'P'C'$ contains at least one point and is a subset of M. Similarly, we may show the existence of an arc $X''P''C''$ such that (1) the segment $X''P''C''$ contains at least one point and is a subset of I_1 and (2) the points X'' and C'' are points of the segments AXB and ACB, respectively. Let $X'X''$ denote the point X' or the arc of the segment AXB from X' to X'', according as X'' is or is not identical with X'. Also let $C'C''$ denote the point C' or the arc of the segment ACB from C' to C'', according as C'' is or is not identical with C'. By means of repeated applications of Axiom C it may be shown that the point set $X'P'C' + C'C'' + X''P''C'' + X'X''$ contains a simple closed curve J', which contains at least one point of each of the segments $X'P'C'$ and $X''P''C''$. Let I' denote the interior of J'. Now I' is a subset of I by Theorem 1. Thus the segment ADB is a subset of E', the exterior of J'. Since the connected set I_2 plus the segment ADB contains no point of J' but does contain a point of E', it follows that I_2 plus the segment ADB is a subset of E'. Furthermore I' cannot contain a point of the segment AXB. For, suppose I' contains the point F of the segment AXB. The connected set $F + I_2$ contains no point of J' but contains points of both complementary domains of J'. We conclude from this contradiction that the segment AXB is a subset of $J' + E'$. Now I' cannot be a subset of M since there exists a point of $J' \cdot I_1$ which is either a limit point of I' or contiguous to a point of I', and M and I_1 are two mutually separated domains. Also, I' cannot be a subset of I_1 since there exists a point of $J'\cdot M$ which is either a limit point of I' or contiguous to a point of I', and M and I_1 are two mutually separated domains. Also I' cannot be a subset of $M+I_1$ and contain points of both M and I_1 since I' would thus be the sum of two mutually separated sets, contrary to Axiom 3. Thus in any case we reached a contradiction, and the theorem is established.

THEOREM 4. *If the points A and B separate the points C and D on the simple closed curve J, and if the segments AXB and CYD are both subsets of I, the interior of J, then these segments have at least one point in common.*

Proof. Suppose the segments AXB and CYD have no point in common. There exists an arc $A'\omega$ such that $A'\omega-A'$ is a subset of E, the exterior of J, and such that A' is a point of the segment CAD of J. Similarly, there exists an arc $B'\omega$ such that $B'\omega-B'$ is a subset of E and B' is a point of the segment CBD. The point set $A'\omega+B'\omega$ contains an arc $A'X'B'$ such that the segment $A'X'B'$ is a subset of E. Let AA' denote the point A or the arc of the segment CAD from A to A' according as A' is or is not A, and let BB' denote the point B or the arc of the segment CBD from B to B' according as B' is or is not B. It may be shown that there exists a simple closed curve J^* satisfying the following conditions: (1) J^* is a subset of $AA'+BB'+AXB+A'X'B'$, (2) J^* contains at least one point of each of the segments AXB and $A'X'B'$, and (3) $J^*\cdot J$ is the sum of two mutually separated continua which separate C and D on J; therefore $J-J^*\cdot J=g_1+g_2$, where g_1 and g_2 are mutually separated segments containing C and D, respectively. Since J^* contains no point of the arc CYD, it follows that the connected point set $CYD+g_1+g_2$ is a subset of I^*, a complementary domain of J^*. Thus J is a subset of I^*+J^*. Hence, by Theorem 1, either I or E is a subset of I^*. But each of the sets I and E contains a point of J^*, and J^* contains no point of I^*. Thus we reach a contradiction, and the theorem is established.

THEOREM 5. *If J is a simple closed curve or triune, then I, the interior of J, contains infinitely many points.*

Proof. Suppose I contains exactly n points, where n denotes a natural number. Thus every point of J is contiguous to some point of I. If there are infinitely many points of J, the closed point set I contains a limit point of J by Axiom C. But this is impossible since J is closed. Hence J contains only a finite number of points. Let A and B be two contiguous points of J, and let X_0 be a point of $J-(A+B)$. Let X be a point of I which is contiguous to A and Y a point of I contiguous to B. Let XY denote the point X or an arc from X

to Y lying in I, according as Y is or is not identical with X. The point set $XY+A+B$ contains a simple closed curve or triune J_1 which is a subset of $J+I$ and contains A, B, and a point X_1 of I but is such that X_0 is in the exterior of J_1. Similarly, there exists a simple closed curve or triune J_2 which is a subset of J_1 plus its interior and contains A, B, and a point X_2 of the interior of J_1 but is such that X_1 is in the exterior of J_2. By continuing the indicated process $n+1$ times we reach a contradiction, since $X_1, X_2, X_3, \cdots, X_{n-1}$ are distinct points of I. Hence I must contain infinitely many points.

THEOREM 6. *Let J denote a simple closed curve whose interior I contains a point P which is contiguous to at least three distinct points of J. Let $P_1, P_2, \cdots, P_n$, $(n \geqq 3)$, be points of J (in the order indicated if $n>3$), and let $\beta = P_1+P_2+ \cdots +P_n$. Suppose β is the set of all points of J that are contiguous to P. Let P_kP_{k+1}, $(k=1, 2, \cdots, n)$, denote that arc of J which contains only the points P_k and P_{k+1} of the set β, and let P_{n+1} denote P_1. Let I_k, $(k=1, 2, \cdots, n)$, denote the interior of the triune or simple closed curve $P+P_kP_{k+1}$. Then $I=P+\sum_{k=1}^{n} I_k$.*

Proof. By Theorem 1, I_k, $(k=1, 2, \cdots, n)$, is a subset of I. Also, if $k \neq j$, then I_k and I_j are mutually exclusive. Suppose $I-(P+\sum_{k=1}^{n} I_k)=M$, where M is a non-vacuous point set, and let M_1 denote a component of M. Now M is a domain; hence M_1 is a domain. Let α denote the set of all points of J, each of which is either a limit point of M_1 or contiguous to a point of M_1. There exists at least one point of α. For let Q_1 denote any point of M_1, and let $Q_1\omega$ denote an arc from Q_1 to ω lying in the exterior of the simple closed curve or triune $P+P_1P_2$. Let Q_1' denote the first point that $Q_1\omega$ has, in order Q_1 to ω, in common with J. Now Q_1' is obviously a point of α, since Q_1' is the first point of $Q_1\omega$, in order Q_1 to ω, which does not belong to M_1.

Suppose there exist two points X and Y of α, which do not lie together on one of the arcs $P_1P_2, P_2P_3, \cdots, P_nP_1$. Since X and Y are non-contiguous points of J, it follows that J is the sum of two arcs from X to Y having nothing in common except their end points such that the corresponding segments of these arcs are mutually separated. There exists a point P_r of β on one of these segments and a point P_s of β on the other. Since P_r and P_s are both points of β, P_r+P+P_s is an arc lying in $I+J-(M+X+Y)$. There exists an arc P_rQP_s which is a subset of $J+E-(X+Y)$, where E denotes the exterior of J, and is such that $P_rQP_s \cdot J$ is the sum of two mutually separated connected point sets M_r and M_s which contain P_r and P_s, respectively, but no other points of β, and which separate X and Y on J. The point set $P+P_rQP_s$ is a simple closed curve J' which contains P and is such that $J' \cdot J$ is the sum of two mutually separated connected point sets which separate X and Y on J.

Thus $J-J'\cdot J$ is the sum of two mutually separated segments g_X and g_Y which contain X and Y, respectively. Now J' separates X from Y, for otherwise g_X and g_Y would lie together in I', a complementary domain of J'; therefore by Theorem 1, either I or E would be a subset of I', which is contrary to the fact that both I and E contain points of J'. Since X and Y are boundary points of M_1 lying in different complementary domains of J', it follows that there are points of M_1 lying in different complementary domains of J'. This is impossible since M_1 contains no point of J'. From this contradiction we conclude that if X and Y are any two distinct points of α, then there exists an arc of the set $P_1P_2, P_2P_3, \cdots, P_nP_1$ which contains both X and Y.

Thus, if $J-\beta$ contains a point X of α, then the arc of the set P_1P_2, $P_2P_3, \cdots, P_nP_1$ which contains X must contain all of α. Now if α is not a subset of one of the arcs of the set $P_1P_2, P_2P_3, \cdots, P_nP_1$, it follows that $J-\beta$ contains no point of α. Furthermore, it follows that $n=3$ and $\alpha=\beta=P_1+P_2+P_3$. In this case, since J is not a triune, one of the segments P_1P_2, P_2P_3, P_3P_1 exists. Suppose the segment P_3P_1 is non-vacuous. Using methods similar to those used in the early part of the proof to get J', we obtain a simple closed curve J^* satisfying the following conditions: (1) J^* is a subset of $P+J+I_3+E$, (2) J^* contains P, P_2, a point of the segment P_3P_1, a point of E, and a point of I_3, and (3) $J^*\cdot J$ is the sum of two mutually separated connected point sets which separate P_1 and P_3 on J, and $J-J^*\cdot J$ is therefore the sum of two mutually separated segments, one containing P_1 and the other containing P_3. The argument used above to show that J' separates X and Y may be applied here to show that J^* separates P_1 and P_3. Hence J^* separates two points P_1' and P_3' of M_1, since P_1 and P_3 are points of α. But this is impossible since M_1 is connected and contains no point of J^*. Thus we reach a contradiction, and we conclude that there exists an arc P_iP_{i+1} of the set of arcs $P_1P_2, P_2P_3, \cdots, P_nP_1$ which contains α.

Let E_i denote the exterior of the simple closed curve or triune $(P+P_iP_{i+1})$. The domain E_i contains M_1, and we write $E_i=M_1+(E_i-M_1)$. The boundary of M_1 is a subset of $(P+P_iP_{i+1})$. Hence M_1 and E_i-M_1 are mutually separated, contrary to Axiom 3. Thus we reach a contradiction, and the theorem is established.

THEOREM 7. *Let $P_1, P_2, \cdots, P_n$, $(n\geqq 3)$, be points of the simple closed curve J (in the order indicated if $n>3$). Let $A_1, A_2, \cdots, A_k$, $(k\geqq 2)$, be points of the arc A_1A_k (in the order indicated if $k>2$) where A_1A_k is a subset of I, the interior of J. Let $\beta=P_1+P_2+\cdots+P_n$, and let $\gamma=A_1+A_2+\cdots+A_k$. Suppose β is the set of all points of J each of which is contiguous to at least one point of A_1A_k, and suppose γ is the set of all points of A_1A_k each of which is contiguous to at least one point of β. Suppose furthermore that no point of β is contiguous to both*

A_1 and A_k, but that P_1 is contiguous to A_1 and to no other point of γ and P_n is contiguous to A_k and to no other point of γ. It follows that if $j<j'$, and if P_j is contiguous to A_i and $P_{j'}$ is contiguous to $A_{i'}$, then $i\leqq i'$. Let P_jP_{j+1}, $(j=1, 2, \cdots, n)$, denote that arc of J from P_j to P_{j+1} which contains no other point of β, where P_{n+1} denotes P_1. Let A_iA_{i+1}, $(i=1, 2, \cdots, k-1)$, denote the arc of A_1A_k from A_i to A_{i+1}. Let I_j, $(j=1, 2, \cdots, n-1)$, denote the interior of C_j, where C_j denotes the triune or simple closed curve $(P_jP_{j+1}+A_i)$ in case P_j and P_{j+1} are both contiguous to A_i, or C_j denotes the simple closed curve $(P_jP_{j+1}+A_iA_{i+1})$ in case no point of γ is contiguous to both P_j and P_{j+1}, but where P_j is contiguous to A_i and P_{j+1} is contiguous to A_{i+1}. Let I_n denote the interior of the simple closed curve $(P_nP_1+A_1A_k)$. If P_j is contiguous to each of the points $A_{i_j}, A_{i_j+1}, \cdots, A_{i_j+k_j}$, let $I_{j1}, I_{j2}, \cdots, I_{jk_j}$ denote the interiors of $(P_j+A_{i_j}A_{i_j+1})$, $(P_j+A_{i_j+1}A_{i_j+2})$, $\cdots$, and $(P_j+A_{i_j+k_j-1}A_{i_j+k_j})$, respectively. Then

$$I = A_1A_k + \sum_{j=1}^{n} I_j + \sum_{j=1}^{n}\sum_{t=1}^{k_j} I_{jt},$$

where I_{jt} is a null set if P_j is contiguous to only one point of γ.

The theorem is illustrated by Fig. 1.

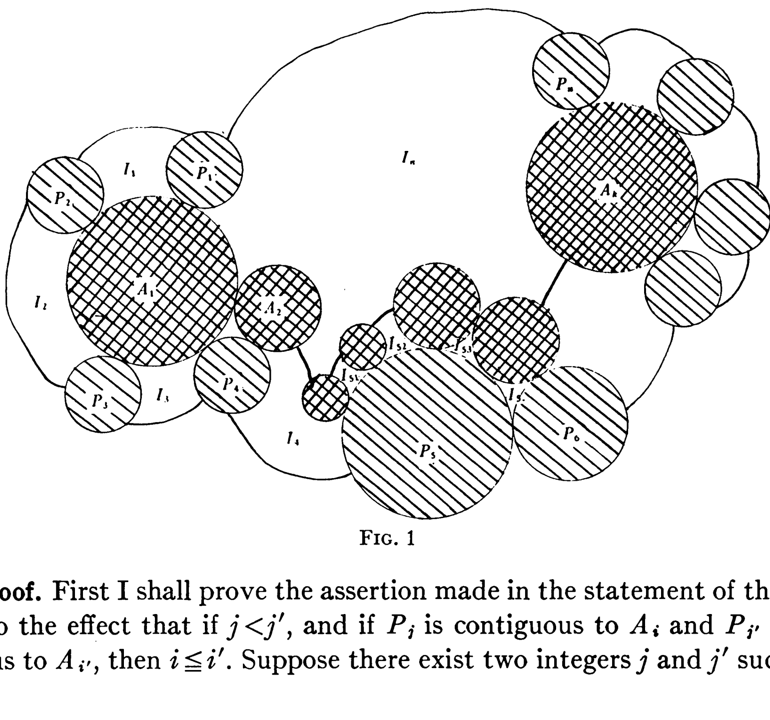

FIG. 1

Proof. First I shall prove the assertion made in the statement of the theorem to the effect that if $j<j'$, and if P_j is contiguous to A_i and $P_{j'}$ is contiguous to $A_{i'}$, then $i\leqq i'$. Suppose there exist two integers j and j' such that

$j<j'$, P_j and $P_{j'}$ are contiguous to A_i and $A_{i'}$, respectively, and $i>i'$. It follows that $i>1$ and $i'<k$. Also since P_1 is contiguous to A_1 but to no other point of γ, we see that $1<j$. Again, since P_n is contiguous to A_k but to no other point of γ, we see that $j'<n$. Thus $1<j<j'<n$; hence P_1 and $P_{j'}$ separate P_j and P_n on J. Let $A_{i_1'}$ be the point of γ with lowest subscript which is contiguous to $P_{j'}$, and let A_{i_1} be the point of γ with the highest subscript which is contiguous to P_j. Let $A_1A_{i_1'}$ denote the arc $A_1A_{i_1'}$ of A_1A_k or the point A_1 according as i_1' is not or is 1, and let $A_{i_1}A_k$ denote the arc $A_{i_1}A_k$ of A_1A_k or the point A_k according as i_1 is not or is k. Now $P_1+A_1A_{i_1'}+P_{j'}$ and $P_j+A_{i_1}A_k+P_n$ are arcs satisfying the conditions of Theorem 4. Hence $A_1A_{i_1'}$ and $A_{i_1}A_k$ have a point in common. But $1\leqq i_1'\leqq i'<i\leqq i_1\leqq k$; therefore $A_1A_{i_1'}$ and $A_{i_1}A_k$ cannot have a point in common. Thus we reach a contradiction, and we conclude that if $j<j'$, and if P_j and $P_{j'}$ are contiguous to A_i and $A_{i'}$, respectively, then $i\leqq i'$.

This result enables us to verify the following implications made in the statement of the theorem: (1) there do not exist two distinct points of γ each of which is contiguous to both P_j and P_{j+1}; (2) if no point of γ is contiguous to both P_j and P_{j+1}, then there exists an integer i such that A_i is contiguous to P_j and A_{i+1} is contiguous to P_{j+1}; and (3) if no point of β is contiguous to both A_i and A_{i+1}, then there exists an integer j such that $j<n$ and P_j and P_{j+1} are contiguous to A_i and A_{i+1}, respectively.

Suppose $I-(A_1A_k+\sum_{j=1}^{n}I_j+\sum_{j=1}^{n}\sum_{t=1}^{k_j}I_{jt})=M$ is a non-vacuous point set. Let M_1 be a component of the domain M, and let α denote the boundary of M_1. Obviously α is a subset of $J+A_1A_k$. Furthermore, α contains at least one point of J and at least one point of A_1A_k. For let P be a point of M_1, and let $P\omega$ be an arc from P to ω lying in the exterior of $(A_1A_k+P_nP_1)$. Now $P\omega$ obviously contains a point of $\alpha\cdot J$. Similarly, let PA_1 denote an arc from P to A_1 which lies in I. Now PA_1 contains a point of $\alpha\cdot A_1A_k$.

Suppose J contains two non-contiguous points X and Y of α. If X and Y do not lie together on any one of the arcs P_1P_2, P_2P_3, $\cdots$, P_nP_1, there exist two points P_r and P_s of β which separate X and Y on J; hence there exists a simple closed curve J' such that: (1) J' is a subset of $E+J+A_1A_k$; (2) $J'\cdot J=M_r+M_s$ where M_r and M_s are two mutually separated continua which contain P_r and P_s, respectively, and which separate X and Y on J; therefore $J-J'\cdot J=g_X+g_Y$ where g_X and g_Y are mutually separated segments containing X and Y, respectively; (3) J' contains at least one point O of E and at least one point of A_1A_k. If X and Y lie together on the arc P_jP_{j+1} of the set P_1P_2, P_2P_3, $\cdots$, P_nP_1, there exists a simple closed curve J' such that: (1) J' is a subset of E plus the segment XY of $P_jP_{j+1}+I_j+A_1A_k+(J-P_jP_{j+1})$; (2) $J'\cdot J$ is the sum of two mutually separated continua which separate X

and Y on J; hence $J-J'\cdot J=g_X+g_Y$ where g_X and g_Y are mutually separated segments containing X and Y, respectively; (3) J' contains at least one point O of E and at least one point of A_1A_k. Now, in either case it may be readily shown that J' separates X from Y and hence that J' separates two points X' and Y' of M_1. But this is impossible since M_1 is connected and contains no point of J'. From this contradiction we conclude that $\alpha\cdot J$ consists either of a single point or of two contiguous points. By an analogous argument it may be shown that $\alpha\cdot A_1A_k$ consists either of a single point or of two contiguous points.

Now I wish to show that α is a subset of a simple closed curve or triune J^*, which is a subset of $J+A_1A_k$, satisfying the condition that either there exists an integer j such that I_j is the interior of J^*, or there exists a pair of integers (j, t) such that I_{jt} is the interior of J^*. If $\alpha\cdot J$ is a subset of P_nP_1, then $J^*=P_nP_1+A_1A_k$. If $\alpha\cdot J$ is not a subset of P_nP_1, we shall use the following procedure.

Case 1. Suppose that P_jP_{j+1}, $(j\neq n)$, is the only arc of the set P_1P_2, $P_2P_3, \cdots, P_nP_1$ which contains $\alpha\cdot J$. It follows that if X denotes either P_j or P_{j+1}, then $P_jP_{j+1}-X$ contains at least one point Z of α. If P_j and P_{j+1} are both contiguous to the point A_i of γ, then we may show that $A_i=\alpha\cdot A_1A_k$. Suppose the contrary. Let Y denote a point of $\alpha\cdot A_1A_k$ which is different from A_i. Now Y either precedes or follows A_i, in the order A_1 to A_k. Suppose Y precedes A_i. Then $i\neq 1$ and $j\neq 1$, and Y is not contiguous to P_{j+1}. There exists a simple closed curve J' having the following properties: (1) J' is a subset of $J+A_1A_k+I_n$; (2) $J'\cdot J$ is a connected point set such that the connected point set $J-J'\cdot J$ contains $P_n+(P_jP_{j+1}-P_j)$; (3) $J'\cdot A_1A_k$ is a connected set such that $A_1A_k-J'\cdot A_1A_k$ is either a connected set containing Y and A_1, or the sum of two mutually separated connected sets, one containing Y and A_1 and the other containing A_k; (4) J' contains at least one point of I_n and at least one point of $J-P_nP_1$. By means of these four conditions imposed on J' we may readily show that J' separates Y from Z and hence that J' separates two points Y' and Z' of M_1. But this is impossible since M_1 is connected and contains no point of J'. Thus we conclude that Y cannot precede A_i on A_1A_k. Similarly Y cannot follow A_i on A_1A_k. Therefore $A_i=\alpha\cdot A_1A_k$. Thus $(A_i+P_jP_{j+1})$ is a simple closed curve or triune J^* which contains α and whose interior is I_j.

If there exists no point of γ which is contiguous to both P_j and P_{j+1}, then there exists an integer i such that P_j is contiguous to A_i and P_{j+1} is contiguous to A_{i+1}. In this case we show that $\alpha\cdot A_1A_k$ is a subset of A_iA_{i+1}. Suppose the contrary. Let Y denote a point of $\alpha\cdot A_1A_k$ not on A_iA_{i+1}. Either Y precedes A_i or Y follows A_{i+1}, in the order A_1 to A_k. Suppose Y precedes A_i.

Then $i \neq 1$ and $j \neq 1$ or n. Now there exists a simple closed curve J' having the following properties: (1) J' is a subset of $J+A_1A_k+I_n$; (2) $J' \cdot J$ is a connected point set such that the connected set $J-J' \cdot J$ contains $P_n+(P_jP_{j+1}-P_j)$; (3) $J' \cdot A_1A_k$ is connected, and $A_1A_k - J' \cdot A_1A_k$ is the sum of two mutually separated connected point sets, one containing Y and A_1, and the other containing A_{i+1} and A_k; (4) J' contains at least one point of I_n and at least one point of $J-P_nP_1$. Thus again J' separates Y from Z and hence separates two points Y' and Z' of M_1. Again we reach a contradiction, and we conclude that Y cannot precede A_i. The same argument applies to show that Y cannot follow A_{i+1}. Hence A_1A_k is a subset of A_iA_{i+1}, and $(P_jP_{j+1}+A_iA_{i+1})$ is therefore a simple closed curve J^* containing α. The interior of J^* is I_j.

Case 2. If there exists no integer r such that P_rP_{r+1} is the only arc of the set $P_1P_2, P_2P_3, \cdots, P_nP_1$ which contains $\alpha \cdot J$, it follows that $\alpha \cdot J$ is a point P_j of β. The case in which $\alpha \cdot J = P_n$ has been disposed of in the paragraph preceding Case 1. Hence suppose $j<n$. Let A_i be the point of γ with lowest subscript which is contiguous to P_j, and let A_i' be the point of γ with highest subscript which is contiguous to P_j. By means of an argument like that used in Case 1, it may be shown that no point of $\alpha \cdot A_1A_k$ precedes A_i or follows A_i' in the order from A_1 to A_k. If $i=i'$, then $\alpha \cdot A_1A_k = A_i$. Thus $(P_jP_{j+1}+A_i)$ or $(P_jP_{j+1}+A_iA_{i+1})$, according as P_j and P_{j+1} are or are not both contiguous to A_i, is a triune or simple closed curve J^* which contains α and whose interior is I_j. If $i \neq i'$, then the arc A_iA_i' of A_1A_k contains $\alpha \cdot A_1A_k$. Furthermore, since $\alpha \cdot A_1A_k$ consists either of a single point or of two contiguous points, it follows that $\alpha \cdot A_1A_k$ is a subset of one of the arcs of the set $A_iA_{i+1}, A_{i+1}A_{i+2}, \cdots, A_{i'-1}A_{i'}$. Suppose the arc $A_{i+t-1}A_{i+t}$ contains $\alpha \cdot A_1A_k$. Then since A_s, $(i \leqq s \leqq i')$, is contiguous to P_j, it follows that $(P_j+A_{i+t-1}A_{i+t})$ is a simple closed curve or triune J^* which contains α and whose interior is I_{jt}. Thus for any case it has been shown that α is a subset of a simple closed curve or triune J^* satisfying the condition that either there exists an integer j such that I_j is the interior of J^*, or there exists a pair of integers (j, t) such that I_{jt} is the interior of J^*.

Let E^* denote the exterior of J^*. Now E^* contains M_1; hence $E^* = M_1 + (E^* - M_1)$, where both M_1 and $E^* - M_1$ are non-vacuous point sets neither of which contains a point of α. Thus E^* is the sum of two mutually separated sets, contrary to Axiom 3; and the theorem is established.

THEOREM 8. *Let the following changes, and none other, be made in the hypotheses of Theorem* 7: $A_1, A_2, \cdots, A_k$, $(k \geqq 1)$, *are points of an arc* A_1T (*in the order indicated if* $k>1$) *where* $A_1T - T$ *is a subset of* I *and* T *is a point of* *the segment* P_nP_1, $(n \geqq 2)$. *The set* $\gamma = A_1 + A_2 + \cdots + A_k$ *is the set of all points of* $A_1T - T$ *which are contiguous to points of* J; $\beta = P_1 + P_2 + \cdots + P_n$ *is the set of all points of* J *which are contiguous to points of* γ. *The point* P_n *is no longer restricted to be contiguous to* A_k *alone of the set* γ; I_n *denotes the interior of* $P_nT + A_kT$; *and* I_0 *denotes the interior of* $A_1T + TP_1$. *Then*

$$I = (A_1T - T) + \sum_{j=0}^{n} I_j + \sum_{j=1}^{n} \sum_{t=1}^{k_j} I_{jt}.$$

If $\beta = P_1 + P_2 + P_3 + \cdots$ *and* $\gamma = A_1 + A_2 + A_3 + \cdots$, *where* β *and* γ *are infinite sets each having* T *as its only limit point, then*

$$I = (A_1T - T) + \sum_{j=0}^{\infty} I_j + \sum_{j=1}^{\infty} \sum_{t=1}^{k_j} I_{jt}.$$

THEOREM 9. *Let the following changes, and none other, be made in the hypotheses of Theorem* 7: V, A_1, A_2, $\cdots$, A_k, T, $(k \geqq 1)$, *are points of the arc* VA_1T *in the order indicated, where the segment* VA_1T *is a subset of* I, *and where* V *and* T *are points of* J *in the order* V, P_1, P_2, $\cdots$, P_n, T, $(n \geqq 1)$. *The set* $\gamma = A_1 + A_2 + \cdots + A_k$ *is the set of all points of the segment* VA_1T *which are contiguous to points of* J, *and* $\beta = P_1 + P_2 + \cdots + P_n$ *is the set of all points of* J *which are contiguous to points of* γ. *The points* P_1 *and* P_n *are no longer restricted as to the number of points of* γ *to which they are contiguous. If* I_0, I_n, *and* I^* *denote the interiors of* $(VP_1 + VA_1)$, $(P_nT + A_kT)$, *and* $(TV + VA_1T)$, *respectively, where* TV *is that arc of* J *not containing* P_1, *then*

$$I = \textit{segment } VA_1T + I^* + \sum_{j=0}^{n} I_j + \sum_{j=1}^{n} \sum_{t=1}^{k_j} I_{jt}.$$

If $\beta = P_1 + P_2 + P_3 + \cdots$ *and* $\gamma = A_1 + A_2 + A_3 + \cdots$ *are infinite sets each having* T *as its only limit point, then*

$$I = \textit{segment } VA_1T + I^* + \sum_{j=0}^{\infty} I_j + \sum_{j=1}^{\infty} \sum_{t=1}^{k_j} I_{jt}.$$

A similar formula holds for the case in which each of the points V and T is approached sequentially by a sequence from β and by a sequence from γ.

THEOREM 10. *Let the following changes be made in the hypotheses of Theorem* 7: P_1 *is contiguous to both* A_1 *and* A_k *but to no other point of* γ. *Either each of the points* A_1 *and* A_k *is contiguous to some point of* $\beta - P_1$ *or else* $k > 2$. *Then either it is true that if* P_j *and* $P_{j'}$ *are contiguous to* A_i *and* $A_{i'}$, *respectively, and if* $j < j'$, *then* $i \leqq i'$, *or it is true that if* P_j *and* $P_{j'}$ *are contiguous to* A_i *and* $A_{i'}$, *respectively, and if* $j < j'$, *then* $i \geqq i'$. *If the former condition holds, if* I^* *denotes* *the interior of* $A_1A_k+P_1$, *and if* I_n *denotes the point set obtained by substituting* n *for* j *in the definition of* I_j *given in the statement of Theorem* 7, *then*

$$I = A_1A_k + I^* + \sum_{j=1}^{n} I_j + \sum_{j=2}^{n}\sum_{t=1}^{k_j} I_{jt}.$$

The theorem is illustrated by Fig. 2.

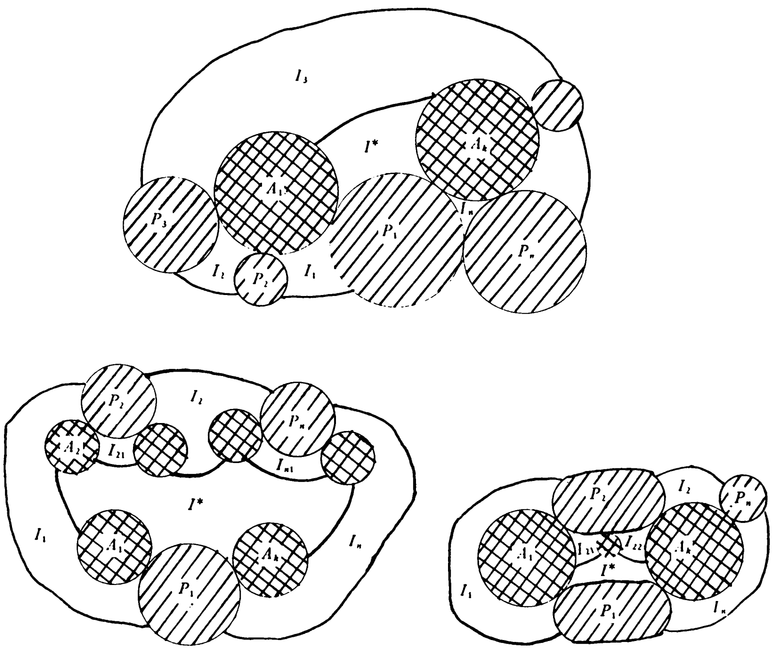

FIG. 2

If in Theorem 10, $\beta=P_1+P_2$, where P_1 and P_2 are non-contiguous, then J is the sum of two arcs, P_1XP_2 and P_1YP_2, and is such that the corresponding segments are mutually separated. Suppose $k\geqq 3$. (1) If at least one of the points A_1 and A_k, say A_1, is contiguous to P_2, then by Theorem 3, $I-A_1=D_1+D_2$, where D_1 and D_2 are the interiors of the simple closed curves $P_1XP_2+A_1$ and $P_1YP_2+A_1$, respectively. Let I_1 denote that domain above which does not contain A_k. Suppose $I_1=D_1$. Let I_2 denote the interior of $P_1YP_2+A_k$ or of $P_1YP_2+A_{k-1}A_k$, according as A_k is or is not contiguous to P_2. (2) If neither A_1 nor A_k is contiguous to P_2, then $I-A_1A_2=D^*+D^{**}$, where D^* and D^{**} are the interiors of $P_1XP_2+A_1A_2$ and $P_1YP_2+A_1A_2$, respectively. Let I_1 denote the domain above which does not contain A_k. Suppose $I_1=D^*$. Let I_2 denote the interior of $P_1YP_2+A_{k-1}A_k$. Then in either of these two cases we have, in accordance with the above notation, the following theorem:

THEOREM 11. $I = A_1A_k + I^* + I_1 + I_2 + \sum_{t=1}^{k_2} I_{2t}$.

This theorem is illustrated by Fig. 3.

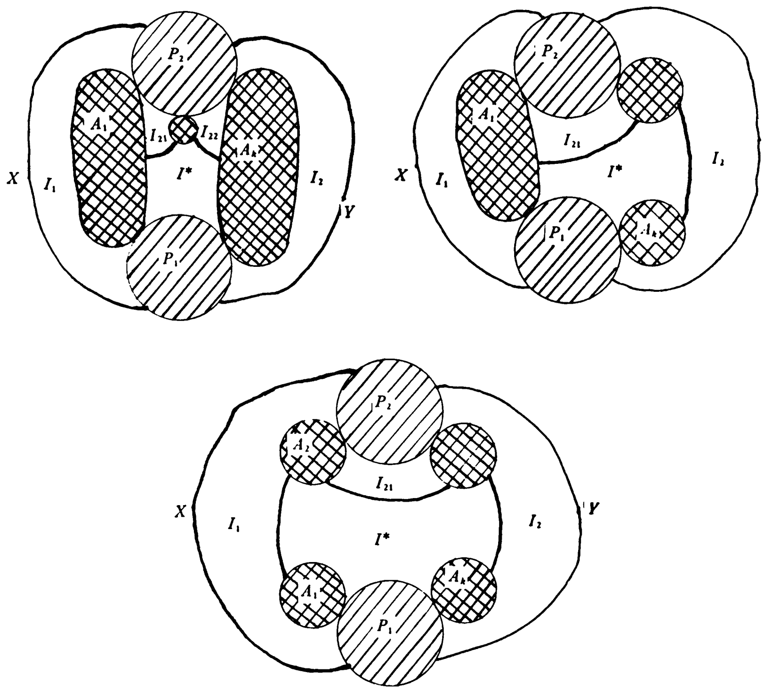

FIG. 3

THEOREM 12. *Let J and C denote two point sets, each of which is either a simple closed curve or a triune. Let I and D denote the interiors of J and C, respectively, and suppose P is a point of $I \cdot D$. Then there exists a simple closed curve or triune Q such that* (1) *Q is a subset of $J + C$, and* (2) *the interior of Q contains P and is a subset of $I \cdot D$.*

Indication of proof. If at least one of the two sets J and C is a triune, one of the sets is a subset of the other plus its interior and hence will have the properties required of Q.

A similar situation exists if both J and C are simple closed curves and one of them is a subset of the other plus its interior.

If neither of the simple closed curves J and C is a subset of the other plus its interior, let A be a point of $J \cdot D$, and let PA be an arc from P to A lying wholly in D. Let O denote the first point, in the order P to A, that PA has in common with J. I wish to show the existence of a simple closed curve or triune Q' such that (1) Q' is a subset of $C+J\cdot D$, (2) $Q'\cdot C$ is connected, (3) $Q'\cdot J\cdot D$ is connected and contains O, and (4) the interior of Q' contains $(PO-O)$. Obviously, the existence of Q' will be established if it can be shown that there exists a point set K (which shall contain $Q'\cdot J\cdot D$) which is a connected subset of J containing O and which satisfies with respect to C those conditions satisfied by any one of the point sets P, A_1A_k, segment AXB, (A_1T-T), segment VA_1T, A_1A_k, or A_1A_k with respect to the corresponding given simple closed curve of Theorems 6, 7, 3, 8, 9, 10, and 11, respectively. There are several cases.

Case 1. Suppose O is not contiguous to any point of C. Let B denote a point of $J\cdot E$. The set J is the sum of two arcs from O to B, say OXB and OYB. Let M denote the closed point set consisting of C together with all points Z of D such that Z is contiguous to at least one point of C. Let A_1 and A_2 denote the first points that OXB and OYB have in common with M, in the order O to B. If both A_1 and A_2 are points of C, the hypothesis of Theorem 3 is satisfied, and the segment A_1OA_2 is the desired point set K.

Case 2. Suppose O is not contiguous to any point of C and that one of the points A_1 and A_2 described in Case 1, say A_1, is a point of D, while the other, A_2, is a point of C. If A_1 is contiguous to two or more points of C, then the hypotheses of Theorem 8 are satisfied, and $(A_1OA_2-A_2)$ is the desired point set K. If A_1 is contiguous to only one point F of C, and F is not contiguous to A_2, then the hypotheses of Theorem 3 are satisfied, and $(A_1OA_2-A_2)$ is the desired point set K. If F is contiguous to A_2, let A_1' denote the first point of OXB, in the order O to B, which is either a point of D that is contiguous to a point of $C-F$ or else a point of C. Now if A_1' is a point of D, the hypothesis of Theorem 8 is satisfied, and $(A_1'OA_2-A_2)$ is the desired point set K. If A_1' is a point of C, the hypothesis of Theorem 9 is satisfied, and the segment A_1OA_2 is the desired point set K.

The remaining cases may be treated with similar methods.

If Q' is a triune, Q' has the properties required of Q, since the interior of Q' cannot contain any points of J. (No simple closed curve may contain a point of each complementary domain of a triune.)

If Q' is a simple closed curve whose interior contains no point of J, then $Q'=Q$. If the interior of Q' contains any points of J, let M denote the set of all such points of J. Now, $M+Q'$ contains a simple closed curve Q having the required properties. The proof of this last statement is little different from the proof for spaces in which there do not exist contiguous points.

As an immediate result of Theorem 12 we have the following theorem:

THEOREM 13. *Let $J_1, J_2, \cdots, J_n$ denote n point sets each of which is either a simple closed curve or a triune; and suppose that $I_1, I_2, \cdots, I_n$, the interiors of $J_1, J_2, \cdots, J_n$, respectively, have a point O in common. Then there exists a simple closed curve or triune J which is a subset of $J_1+J_2+\cdots+J_n$ and is such that I, the interior of J, is a subset of $I_1 \cdot I_2 \cdot \cdots \cdot I_n$ and contains O.*

THEOREM 14. *Suppose each of the sets J_1 and J_2 is either a simple closed curve or a triune. Let I_1 and I_2 denote the interiors of J_1 and J_2, respectively, and suppose I_1+I_2 is connected. Then there exists a simple closed curve or triune J which is a subset of J_1+J_2 and whose interior is a subset of $I_1 \cdot I_2$.*

Indication of proof. Assume the theorem false. Then by Theorem 12 $I_1 \cdot I_2=0$. Hence there exist points P_1 and P_2 of I_1 and I_2, respectively, such that P_1 is contiguous to P_2. Thus P_1 is a point of J_2 and P_2 is a point of J_1. There exists a point F of $J_1 \cdot J_2$ and an arc P_1F of J_2 such that P_1F-F is a subset of I_1. Let P_2F denote an arc of J_1. The point set P_1F+P_2F contains a simple closed curve or triune J' containing P_1 and P_2, whose interior I' is a subset of I_1. The set $J'+I' \cdot J_2$ contains a simple closed curve or triune J satisfying the required conditions. Thus the assumption is false, and the theorem is established.

With the help of Theorems 12 and 14, the following theorem may be established:

THEOREM 15. *Let J_1 and J_2 denote two point sets, each of which is either a simple closed curve or a triune. Let I_1 and I_2 denote the interiors of J_1 and J_2, respectively. Suppose I_1+I_2 is connected. Then there exists a simple closed curve or triune J which is a subset of J_1+J_2, and whose interior contains I_1+I_2.*

THEOREM 16. *Let $J_1, J_2, \cdots, J_n$ denote n point sets each of which is either a simple closed curve or a triune. Let $I_1, I_2, \cdots, I_n$ denote the interiors of $J_1, J_2, \cdots, J_n$, respectively. Suppose $I_1+I_2+\cdots+I_n$ is connected. Then there exists a point set J which is either a simple closed curve or a triune, which is a subset of $J_1+J_2+\cdots+J_n$, and which is such that I, the interior of J, contains $I_1+I_2+\cdots+I_n$.*

THEOREM 17. *If J is a simple closed curve or triune, then I, the interior of J, contains at least one point which is not contiguous to any point of J.*

Proof. Suppose each point of I is contiguous to some point of J. Let P be any point of I. I wish to show the existence of two distinct points of I, each contiguous to P. Since I is connected and contains infinitely many points, P is either contiguous to a point of I or is a limit point of I. By Axiom C, J contains all limit points of I; hence P must be contiguous to a point Q of I. In case there exists a point P_1 of J which is contiguous to both

P and Q, then I_1, the interior of the triune PQP_1, is a connected subset of I. Since J contains all the limit points of I_1, it follows that P is contiguous to a point T of I_1. Thus for this case there exist two distinct points (Q and T) of I, each contiguous to P. In case no point of J is contiguous to both P and Q, let P_1 and Q_1 denote points of J which are contiguous to P and Q, respectively, and let P_1Q_1 denote an arc of J. The point set P_1Q_1+P+Q contains a simple closed curve J_2, which contains P and Q. Let I_2 denote the interior of J_2. The argument used above may be used here to show that P is contiguous to a point T of I_2.

Let QT denote an arc lying in I. The point set $QT+P$ contains a simple closed curve or triune J' whose interior I' contains infinitely many points, no one contiguous to any point of J. Thus we reach a contradiction, and the theorem is established.

Example 1. In the euclidean plane let C_1, C_2, and C_3 denote three circles, each tangent externally to each of the others. Denote their centers by P_1, P_2, and P_3, respectively, and their radical center by O. Let $A_1=C_1\cdot C_2$, $A_2=C_2\cdot C_3$, and $A_3=C_3\cdot C_1$. Let B_1, B_2, and B_3 denote points on the rays OA_1, OA_2, and OA_3, respectively, such that $d(O, B_1)=d(O, B_2)=d(O, B_3)$, and $d(O, B_1)>d(O, A_1)$. Let B_3B_1, B_1B_2, and B_2B_3 be circular arcs having P_1, P_2, and P_3, respectively, as centers and lying on the non-O sides of the lines B_3B_1, B_1B_2, and B_2B_3, respectively. Let $\beta_{11}=B_3B_1+OB_3+OB_1$, $\beta_{21}=B_1B_2+OB_1+OB_2$, $\beta_{31}=B_2B_3+OB_2+OB_3$. For each point P of β_{i1} and each j, ($i=1, 2, 3$; $j=1, 2, 3, \cdots$), let Q_{Pij} denote the point of the interval P_iP such that $d(P_i, Q_{Pij})=d(P_i, A_i)+[d(P_i, P)-d(P_i, A_i)]/j$, and let β_{ij} denote the set of all points Q_{Pij}. Let T_i, ($i=1, 2, 3$), denote C_i plus its interior. Let S denote the following collection: (1) T_i, ($i=1, 2, 3$), is an element of S, and (2) each point of the plane not in $T_1+T_2+T_3$ is an element of S. For each positive integer n, let G_n denote the following subsets of S: (1) the interior of each circle of radius $1/n$ or less which neither encloses nor contains a point of $T_1+T_2+T_3$ is an element of G_n; (2) for each pair (i, j), ($i=1, 2, 3$; $j=n, n+1, \cdots$), the set consisting of T_i and all elements of S enclosed by β_{ij} is an element of G_n. If each element of S is called a "point," each element of G_1 is called a "region," and each of the "points" T_1, T_2, and T_3 is "contiguous to" each of the others, then each of the axioms of this paper is non-vacuously satisfied.

Example 2. In a euclidean space let s_1, s_2, s_3, s_4, and s_5 denote the spheres whose equations are $(x-1)^2+y^2+z^2=3/2^2$, $(x+1/2)^2+(y-(3)^{1/2}/2)^2+z^2=3/2^2$, and $(x+1/2)^2+(y+(3)^{1/2}/2)^2+z^2=3/2^2$, $x^2+y^2+(z-(3)^{1/2}/12)^2=3/12^2$, and $x^2+y^2+(z+(3)^{1/2}/12)^2=3/12^2$, respectively. Each of these spheres is tangent externally to each of the others. For each i, ($i=1, \cdots, 5$), let T_i denote s_i

plus its interior. Let S denote the collection defined as follows: (1) For each i, $(i=1, \cdots, 5)$, T_i is an element of S. (2) Let K denote the (x, y)-plane. Each point of the unbounded component of $K-K(T_1+T_2+T_3)$ is an element of S. (3) For each set of three distinct positive integers i, j, and k, such that each integer is less than 6 and at least one integer is either 4 or 5, let M_{ijk} denote the plane which contains the centers of the spheres s_i, s_j, and s_k. Each point of the bounded component of $M_{ijk}-M_{ijk}(T_i+T_j+T_k)$ is an element of S. For each positive integer n, let G_n denote the subsets of S defined as follows: (1) The interior of each circle of radius $1/n$ or less which is a subset of $S\cdot M_{ijk}$ or of $S\cdot K$ is an element of G_n. (2) In each of the sets $S\cdot M_{ijk}$, construct three sequences of segments like the three sequences of segments constructed within the triune of Example 1, and in $S\cdot K$ construct three sequences of segments like those constructed in the exterior of the triune of Example 1. For each pair (i, j), $(i=1, \cdots, 5; j=n, n+1, \cdots)$, the subset of S consisting of T_i together with all elements of $S-(T_1, T_2, \cdots, T_5)$ each of which is enclosed by T_i plus the jth segment of one of the sequences constructed is an element of G_n. If each element of S is called a "point," each element of G_1 is called a "region," and each of the "points" $T_1, \cdots, T_5$ is "contiguous to" each of the others, then all the axioms of this paper are non-vacuously satisfied.

Example 2 shows that Theorem 6 fails to hold if J denotes a triune instead of a simple closed curve.

THEOREM 18. *If J is a simple closed curve, if I is the interior of J, and if H and K are mutually exclusive compact continua lying in $J+I$, then no two points of H separate two points of K on J.*

Proof. Suppose on the contrary that the points A and B of H separate the points C and D of K on J. It follows from Theorem 24 of S.C.P. that K contains an irreducible continuum T from $K\cdot ACB$ to $K\cdot ADB$. By Theorem 28 of S.C.P., $T-T\cdot J$ is a connected set having boundary points C_1 and D_1 in segments ACB and ADB, respectively. It may be readily shown that the component of $S-(J+H)$ which contains $T-T\cdot J$ contains a segment C_2YD_2, where C_2 and D_2 are points of segments ACB and ADB, respectively, and neither point belongs to H. Thus the arc C_2YD_2 contains no point of H and lies, except for its end points, in I.

The above argument may be repeated to show that there exists an arc A_2XB_2 which contains no point of C_2YD_2, where the segment A_2XB_2 lies in I, and A_2 and B_2 separate C_2 and D_2 on J. But this contradicts Theorem 4.

THEOREM 19. *The interior of a simple closed curve or triune is not a subset of any simple closed curve or triune.*

Proof. Let I denote the interior of a simple closed curve or triune J. By Theorem 5, I is not a subset of a triune. Suppose I is a subset of a simple closed curve C. By Theorem 17, there exists a point P of I which is not contiguous to any point of J. Now P is either a limit point of $S-C$ or is contiguous to a point of $S-C$. If P is a limit point of $S-C$, let R be a connected domain containing P, but containing no point of J, and let Q be a point of $R\cdot(S-C)$. If P is not a limit point of $S-C$, let Q be a point of $S-C$ which is contiguous to P. In either case, Q belongs to $I\cdot(S-C)$.

THEOREM 20. *If neither of the contiguous points X and Y separates the point A from the point B, then their sum does not separate A from B.*

Proof. Suppose $S-(X+Y)=S_A+S_B$, where S_A and S_B are mutually separated sets containing A and B, respectively. There exists an arc α from A to B which does not contain Y. The arc α contains X; hence α contains an arc AX. Similarly, there exists an arc AY. The point set $AX+AY-(X+Y)$ is connected and hence lies in S_A. The set $AX+AY$ contains a simple closed curve or triune J which contains X and Y and is a subset of S_A+X+Y. Let Q be any point of $J-(X+Y)$, and let I and E denote the two complementary domains of J. Since Q is a point of S_A and is a boundary point of each of the connected sets I and E, it follows that I and E are subsets of S_A. Hence S is a subset of S_A+X+Y. Thus we reach a contradiction, and the theorem is established.

AXIOM 4. *If P and Q are two distinct non-contiguous points, there exists a simple curve or triune which separates P from Q.*

With the help of Axiom 4, the Borel-Lebesgue theorem (Theorem 5 of S.C.P.), and Theorem 16, the next theorem may be established.

THEOREM 21. *If H and K are two mutually separated compact continua, there exists a simple closed curve or triune which separates H from K.*

THEOREM 22. *If J is a simple closed curve or triune containing the contiguous points A and B, I is the interior of J, and P is a point of $J+I$ which is not contiguous to A, then there exists a simple closed curve or triune J^* satisfying the following conditions:* (1) *It is a subset of $J+I$.* (2) *Its intersection with J is an arc containing A and B.* (3) *Its exterior contains P.*

Proof. Suppose the theorem is false. If may be readily shown that there exists a simple closed curve or triune which satisfies conditions (1) and (2). Furthermore it may be readily shown that if P is any point of J not contiguous to A, or if P is a point of I not contiguous to A such that there is a simple closed curve or triune J' containing P and satisfying conditions (1) and (2), then there exists a simple closed curve or triune satisfying conditions (1), (2), and (3). Hence our supposition implies that P is a point of I, and every simple closed curve or triune, satisfying conditions (1) and (2), encloses P. Let PX denote an arc from P to some point X of $J-(A+B)$ such that $PX-X$ is a subset of I. Let S_1 denote the set consisting of P and all points Y of $PX-X$ such that every simple closed curve or triune satisfying conditions (1) and (2) encloses the interval PY of PX, and let S_2 denote $PX-S_1$. By the Dedekind-cut proposition (P.S.T., chap. 1, Theorem 64) there exists a point Q which is either the last point of S_1 or the first point of S_2, in the order P to X.

Suppose Q is the first point of S_2. There exists a simple closed curve or triune J' satisfying the following conditions: If $Q=X$, then $J'=J$; if $Q\neq X$, then J' contains Q, satisfies conditions (1) and (2), and encloses $PQ-Q(=S_1)$. There exists a third simple closed curve or triune J_1, which is a subset of J' plus its interior and is such that $J'\cdot J_1$ is an arc containing A and B but not containing Q. Thus Q is exterior to J_1; hence $PQ-Q$ either intersects J_1 or is exterior to J_1. Both these possibilities are ruled out since J_1 satisfies conditions (1) and (2) and hence encloses $PQ-Q$. Thus Q is not the first point of S_2.

Suppose Q is the last point of S_1. By means of an argument analogous to that used in the last paragraph, it may be shown that Q is not contiguous to a point of S_2. Thus Q is a limit point of S_2. Now Q cannot be contiguous to both A and B; for if such were the case, the triune ABQ would satisfy conditions (1) and (2) and hence would enclose $PQ=S_1$ and, in particular, the point Q. Let C denote a point of the pair (A, B) which is not contiguous to Q, and let $C\omega$ be an arc from C to ω lying in J plus its exterior and containing no point of J which is contiguous to Q. By Theorem 21, there exists a simple closed curve or triune J' which separates Q from $C\omega$. Since Q is a limit point of S_2, it follows that I', the interior of J', contains a segment QW of S_2. There exist a point F of the segment QW, and a simple closed curve or triune J_1 which satisfies conditions (1) and (2) and is such that F is in E_1, the exterior of J_1. Let I_1 denote the interior of J_1. By Theorem 12, there exists a simple closed curve or triune J_2, which is a subset of J_1+J' and whose interior I_2 contains Q and is a subset of $I_1\cdot I'$. There exists a point T of $J_2\cdot I'\cdot J_1$. It may be readily shown that the point set $J_1-T+(J_2-T)$ contains a simple closed curve or triune J^* which satisfies conditions (1) and (2). Furthermore, since I_2+T+E_1 is a connected point set containing Q and ω but no point of J^*, Q is exterior to J^*. Hence Q is not a point of S_1.

Thus we have reached a contradiction and the theorem is established.

THEOREM 23. *The interior of every triune is non-compact and so is the interior of every simple closed curve which contains two contiguous points.*

Proof. Let J denote a simple closed curve or triune containing the contiguous points A and B, and let I denote the interior of J. Suppose that I is compact. It follows that $J+I$ is compact, completely separable, and metric. Let $D_1, D_2, D_3, \cdots$ be a sequence of domains which properly covers $J+I$ and with respect to which $J+I$ is completely separable. Let n_1 denote the smallest integer (which exists in view of Theorems 17 and 22) such that there exists a simple closed curve or triune J_1 which is a subset of $J+I$, and such that $J \cdot J_1$ is an arc containing A and B, and D_{n_1} is exterior to J_1. Let $n_1, n_2, n_3, \cdots$ be an increasing sequence of positive integers satisfying the following conditions: For each integer $k>1$, n_k is the smallest integer greater than n_{k-1} such that there exists a simple closed curve or triune J_k which is a subset of J_{k-1} plus its interior, and such that $J_{k-1} \cdot J_k$ is an arc containing A and B, and D_{n_k} is exterior to J_k. For each integer r, let α_r denote the point $J_r-(A+B)$ in case J_r is a triune. Otherwise, let α_r denote an arc P_rQ_r of $J_r-(A+B)$, where Q_r is either contiguous to B or else $d(Q_r, B)<1/r$, and where P_r is either contiguous to A or else $d(P_r, A)<1/r$. For each integer n let E_n denote the exterior of J_n. For each integer r and each point P of α_r there exists an integer n_P such that E_{n_P} contains P. By the Borel-Lebesgue theorem, there exists a finite collection of these domains covering α_r. The domain of this finite collection with greatest subscript E_s, contains all other domains of the finite collection; therefore E_s contains α_r. Hence $\alpha_r \cdot J_s=0$, and consequently $\alpha_r \cdot \alpha_s=0$. Thus there exists an increasing sequence of integers $r_1, r_2, r_3, \cdots$ such that for each integer $k>0$, $\alpha_{r_k} \cdot \alpha_{r_{k+1}}=0$; hence $\alpha_{r_1}, \alpha_{r_2}, \cdots$ is a sequence of mutually exclusive continua. There exists a subsequence of this sequence which converges to a sequential limiting set L containing A and B. Hence by Theorem 33 of S.C.P., L is a perfect continuum and is therefore uncountable. For each integer n, L is a subset of J_n plus its interior. Hence if L contains a point X other than A or B, then X is contiguous to A or B. But the set of all such points X is at most countable by Theorem 14 of S.C.P. Hence L is countably infinite or finite. We have thus reached a contradiction, and the theorem is established.

THEOREM 24.* *If H and K are two mutually separated, closed and compact point sets containing the points A and B, respectively, then there exists a simple closed curve or triune J which separates A from B such that $J \cdot (H+K)=0$.*

Proof. Let h_A and k_B denote the components of H and K containing A and B, respectively. From Theorem 21 it follows that for each component h of H there exists a simple closed curve or triune J_{hB} which separates h from

* Cf. L. Zoretti, *Sur les fonctions analytiques uniformes*, Journal de Mathématiques Pures et Appliquées, vol. 1 (1905), pp. 9–11.

k_B. Let D_{hB} denote that complementary domain of J_{hB} which contains h. The collection of all such domains D_{hB} covers H; hence there exists a finite collection T of such domains covering H. Let T^* denote the point set which is the sum of all the elements of T. By Theorem 16 there exists a simple closed curve or triune J_{AB} whose interior with respect to B contains that component of T^* which contains A. Furthermore, $(H+k_B)\cdot J_{AB}=0$. For each component k of K let J_{Ak} denote a simple closed curve or triune (which exists in view of the preceding argument) which separates A from k and is such that $(H+K)\cdot J_{Ak}=0$. Let D_{Ak} denote the interior of J_{Ak} with respect to A. The collection of all such domains D_{Ak} covers K; hence there exists a finite collection G of such domains covering K. Let G^* denote the sum of the elements of G. By Theorem 16 there exists a simple closed curve or triune J whose interior with respect to A contains that component of G^* which contains B. Furthermore, $(H+K)\cdot J=0$.

THEOREM 25.† *If H and K are two mutually separated closed and compact point sets, and if neither H nor K separates the point A from the point B, then $H+K$ does not separate A from B.*

Proof. Suppose $H+K$ separates A from B.

Case 1. Suppose H is connected. With the help of Theorem 24, it may be shown that there exists a domain D which contains K but no point of H and is the sum of a finite number of components each of which is bounded by a simple closed curve or triune containing no point of $H+K$. Let AXB denote an arc from A to B containing no point of K, and AYB an arc containing no point of H. Let $J_1, J_2, J_3, \cdots, J_r$ denote the boundaries of those components of D which have points in common with AYB. There are four possibilities: (1) A and B are both in $S-D$, (2) A and B are the same component of D, (3) A and B are in different components of D, (4) A or B is in $S-D$ while the other is in D. In each case it may be readily shown that $(AYB-AYB\cdot D)+J_1+J_2+\cdots+J_r+AXB\cdot D$ contains an arc from A to B.

Case 2. Suppose H has only a finite number of components, $h_1, h_2, \cdots, h_n$. By Case 1, h_1+K does not separate A from B. The sets h_2 and h_1+K are mutually separated; hence by Case 1, $h_2+(h_1+K)$ does not separate A from B. Continuing this process we obtain the fact that $h_n+(h_{n-1}+\cdots+h_1+K)$ does not separate A from B.

Case 3. Suppose H has infinitely many components. Let AXB denote an arc containing no point of K. By Axiom C, at most a finite number of components of K contain points which are contiguous to points of AXB. Let

† Cf. P. Alexandroff, *Sur les multiplicités cantoriennes et le théorème de Phragmèn-Brouwer généralisé*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 183, pp. 722–724.

$k_1, k_2, \cdots, k_n$ be the set of all such components of K. By Case 2 there exists an arc AYB which contains no point of $H+k_1+\cdots+k_n$. If $AYB\cdot K=0$, the theorem is established. Suppose $AYB\cdot K\neq 0$. For each component k of K which intersects AYB, the set $H+AXB+K$ is the sum of two mutually separated sets, one containing A and the other containing k, by Theorem 27 of S.C.P. By Theorem 24 there exists a simple closed curve or triune J_k which separates A from k and is such that $J_k\cdot(H+AXB+K)=0$. Let I_k denote the interior of J_k with respect to A. The collection of all such domains I_k covers the closed point set $AYB\cdot K$; hence there exists a finite subcollection T doing so. Let T^* denote the sum of the elements of T. By Theorem 16, for each component t of T^* there exists a simple closed curve or triune C_t which is a subset of the boundary of t and whose interior with respect to A contains t. Let $C_1, C_2, \cdots, C_s$ be the set of all simple closed curves or triunes thus obtained. The point set $(AYB-AYB\cdot T^*)+C_1+C_2+\cdots+C_s$ contains an arc from A to B which contains no point of $H+K$.

THEOREM 26. *If J is a simple closed curve, I is a complementary domain of J, H and K are two mutually separated subcontinua of J, α and β are the two components of $J-(H+K)$, and C is a simple closed curve which separates H from K, then there exists an arc AXB such that A and B are points of α and β, respectively, and segment AXB is a subset of $I\cdot C$.*

Proof. Suppose the theorem is false. Let ω be a point of $S-(J+I+C)$, which exists by Theorem 19. Let D denote the interior of C. One of the sets H and K, say H, is a subset of D. Obviously, C contains at least one point of each of the sets α, β, and I. Let P, Q, Z, and W denote points of $\alpha\cdot C$, $\beta\cdot C$, H, and K, respectively. Let α' and β' denote continua which are subsets of α and β, respectively, and which contain $\alpha\cdot C$ and $\beta\cdot C$, respectively. Now, $\alpha'+\beta'+I\cdot C$ is the sum of two mutually separated continua M_1 and M_2 containing α' and β', respectively. Let E denote the last point of the arc PZQ, in the order P to Q, which is either a point of M_1 or contiguous to a point of M_1; and let F denote the first point of PZQ which is either a point of M_2 or contiguous to a point of M_2. It may be readily seen with the help of Theorem 4 that if E and F are distinct, then E precedes F in the order P to Q. Suppose first that E and F are distinct non-contiguous points. The interval EF of PZQ contains at least one point of H. Furthermore, there exists an arc OL where L is a point of segment EF and $OL-L$ is a subset of $I-I\cdot C$. By Theorem 12 there exists a simple closed curve J' which is a subset of $J+C$ and contains EF, and whose interior I' is a subset of $I\cdot D$ and contains $OL-L$. Now J' contains no point of K, for otherwise $H+I'+K$ would be a connected set containing no point of C. Thus J' contains an arc $E'F'$, where E' and F'

are points of segments LEW and LFW, respectively, and the segment $E'F'$ is a subset of $I\cdot C$. Thus segment $E'F'$ is a subset of M_1 and also a subset of M_2. From this contradiction, it follows that E and F are identical or contiguous. Also $(E+F)\cdot(M_1+M_2)=0$. Similarly, if T denotes the last point of PWQ which is either a point of M_1 or contiguous to a point of M_1, and if V denotes the first point of PWQ which is either a point of M_2 or contiguous to a point of M_2, then T and V are identical or contiguous. Let C' denote a simple curve which separates M_1 from M_2. Let $P'Q'$ denote an arc such that P' and Q' are points of M_1 and M_2, respectively, and segment $P'Q'$ is a subset of $J+I-(J+I)\cdot(C+E+F+T+V)$. Let G denote a point of segment $P'Q'$ which belongs to C', and let NY denote an arc which is a subset of C' and contains G, where N and Y are points of J and segment NY is a subset of I. Now N and Y do not lie in different components of $J-(\alpha'+\beta')$, for otherwise, $H+K+NY$ would be a connected set containing no point of C. Suppose both N and Y are points of that component of $J-(\alpha'+\beta')$ which contains H. If either N or Y preceded E in the order PZQ, we would have two arcs (one a subset of $C\cdot M_1+E$, the other a subset of $C\cdot M_2+$interval $Q'G$ of arc $P'Q'+$interval NG of arc NY) satisfying the hypothesis of Theorem 4 but not the conclusion of this theorem. Similarly, it may be shown that neither N nor Y follows F. Hence N is identical with Y or contiguous to Y. Again a contradiction is reached, and the theorem is established.

THEOREM 27. *If A, X, B, and Y are points of the simple closed curve J in the order indicated, if I is the interior of J, and if H and K are mutually separated closed and compact subsets of $J+I$ such that $H\cdot AXB=0$ and $K\cdot AYB=0$, then there exists an arc AB lying in $J+I-(H+K)$.*

Proof. Two cases will be considered.

Case 1. Suppose there exists no component T of $H+K$ such that $T+J-(A+B)$ is connected. Let N_1 and N_2 denote continua which are subsets of segments AYB and AXB, respectively, and which contain $H\cdot AYB$ and $K\cdot AXB$, respectively; and let α and β denote the components of $J-(N_1+N_2)$ which contain A and B, respectively. By Theorem 27 of S.C.P., N_1+N_2+H+K is the sum of two mutually separated closed sets M_1 and M_2 containing N_1 and N_2, respectively. Hence by Theorem 24 there exists a simple closed curve C which separates N_1 from N_2 and contains no point of M_1+M_2. Thus C contains no point of $H+K$. By Theorem 26 there exists an arc $A'X'B'$, where A' and B' are points of α and β, respectively; and segment $A'X'B'$ lies in $I\cdot C$. Now $A'X'B'+\alpha+\beta$ contains an arc AB lying in $J+I-(H+K)$.

Case 2. Suppose that for at least one component T of $H+K$, $T+J$

$-(A+B)$ is connected. With the help of Axiom C, it may be shown that there is only a finite number of such components. Either H or K, say H, contains at least one such component. There exists a point Y_1 which is the first point of segment AYB, in order A to B, which is either a point of, or contiguous to a point of, such a component of H.

First suppose there exists at least one point Q of AY_1-A having the property that there is a component T_Q of K such that $Q+T_Q+$segment AXB is connected. Let $Q_1, Q_2, \cdots, Q_n$ denote all such points in the order A to Y_1. Obviously Q_i, $(i=1, 2, \cdots, n)$, is not a point of $H+K$. If the segment AQ_1 of AY_1 contains no point of H, let α_1 denote the interval AQ_1. If segment AQ_1 contains a point of H, it may be shown with the help of Theorem 18 and Case 1 of this theorem that there exists an arc α_1, from A to Q_1, lying in $J+I-(H+K)$. Similarly, if $n>1$, there exists an arc α_i, $(i=2, 3, \cdots, n)$, from Q_{i-1} to Q_i, lying in $J+I-(H+K)$. The set $\alpha_1+\alpha_2+\cdots+\alpha_n$ contains an arc α from A to Q_n. Now let X_2 denote the last point of segment AXB, in order A to B, for which there exists a component T_2 of H, such that $Y_1+T_2+X_2$ is connected. If $Q_n=Y_1$, and if there exists a point P of AXB between X_2 and B for which there is a component T' of K such that $Y_1+T'+P$ is connected, then, denoting Q_n by Q', we have an arc $AQ'=\alpha$ having the following properties: (1) The arc AQ' lies in $J+I-(H+K)$, (2) Q' is a point of segment AYB and does not precede Y_1, (3) there is at least one component T of H for which $[T+\text{segment } AXB+\text{interval } AQ' \text{ (of } AYB) -A]$ is connected, but $T+Q'B-Q'$ is not connected, and (4) there exists no component W of $H+K$ such that $W+AYB-Q'$ is connected. If $Q_n=Y_1$, and if there exists no such point P, let X_1 denote the first point of AXB, such that $Y_1+T_2+X_1$ is connected; or if $Q_k\neq Y_1$, let X_1 denote the first point of AXB for which there exists a component T of H such that Y_1+T+X_1 is connected. In either case, X_1 is a point of segment AXB, and with the help of Theorem 18, Case 1, it may be shown that there exists an arc β from Q_n to X_1 lying in $J+I-(H+K)$. If $X_1\neq X_2$, then by means of an argument like that used in obtaining α, it may be shown that there exists an arc γ, from X_1 to X_2, lying in $J+I-(H+K)$. If $X_1=X_2$, let γ denote X_1. If there exists no point X' of AXB between X_2 and B for which there is a component T of K such that segment $AYB+T+X'$ is connected, then the argument for obtaining α may be used here to obtain an arc δ, from X_2 to B, lying in $J+I-(H+K)$. In this case, $\alpha+\beta+\gamma+\delta$ contains an arc from A to B, and the theorem is established. If there exists a point X', as described, let Q' denote the first point of segment AYB for which exists a component T of K, such that $Q'+T+$segment X_2B is connected. It follows from Theorem 18 and Case 1, that there exists an arc η, from X_2 to Q', lying in $J+I-(H+K)$. In this case, $\alpha+\beta+\gamma+\eta$ contains an arc AQ' having the four properties listed above.

Now suppose there is no point Q as described above. Let X_2 have the same meaning as above, and let γ denote an arc from A to X_2 lying in $J+I-(H+K)$. Again, if there exists no point X', as described above, let δ denote an arc from X_2 to B lying in $J+I-(H+K)$. In this case $\gamma+\delta$ contains an arc from A to B, and the theorem is established. If there exists a point X', as described above, let Q' and η be defined as before. Now $\gamma+\eta$ contains an arc AQ' with properties listed above.

Thus, in any case either there exists an arc AB lying in $J+I-(H+K)$, or else there exists an arc AQ' having the properties listed above. In the latter case, if there exists no point Y' of AYB, between Q' and B, for which there exists a component T of H such that $Y'+T+$segment AXB is connected, then the argument for obtaining α may be used to obtain an arc $Q'B$ lying in $J+I-(H+K)$. In such a case, $AQ'+Q'B$ contains an arc from A to B, and the theorem is established. If there exists a point Y', as described above, let Y_2 denote the first such point in the order Q' to B. Now the argument for obtaining AQ' may be repeated to obtain either an arc $Q'B$ lying in $J+I-(H+K)$ (in which case $AQ'+Q'B$ contains an arc from A to B, and the theorem is established), or an arc $Q'Q''$ of such nature that $AQ'+Q'Q''$ contains an arc AQ'' having the following properties: (1) AQ'' lies in $J+I-(H+K)$, (2) Q'' is a point of segment AYB and does not precede Y_2, (3) there are at least two components of H such that if T denotes either of them, then $[T+$segment $AXB+$interval AQ'' of $AYB-A]$ is connected, but $[T+Q''B-Q'']$ is not connected, and (4) there exists no component W of $H+K$, such that $W+AYB-Q''$ is connected. Since there is only a finite number of components of H such that if T denotes any one of them, $T+J-(A+B)$ is connected, the above process may be repeated until we have obtained either an arc AB lying in $J+I-(H+K)$, or an arc $AQ^{(k)}$ lying in $J+I-(H+K)$, where $Q^{(k)}$ is a point of segment AYB, and there is no point Y' between $Q^{(k)}$ and B for which there is a component T of H such that $Y'+T+$segment AXB is connected. In the latter case, the argument for obtaining α may be used to obtain an arc $Q^{(k)}B$ lying in $J+I-(H+K)$. The set $AQ^{(k)}+Q^{(k)}B$ contains an arc AB lying in $J+I-(H+K)$; and the theorem is established.

THEOREM 28.* *If the common part of the closed and compact point sets H and K is a continuum, and if neither H nor K separates the point A from the point B, and if furthermore, $H-H\cdot K$ and $K-H\cdot K$ are mutually separated sets, then $H+K$ does not separate A from B.*

* Cf. S. Janiszewski, *Sur les coupures du plan faites par des continus*, Prace Matematyczno-Fizyczne, vol. 26, 1913. Also Anna M. Mullikin, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

Proof. Suppose on the contrary that $H+K$ separates A from B. Let $M=H+K$ and $T=H\cdot K$. Let S_1 and S_2 denote the components of $S-M$ which contain A and B, respectively. There exist arcs AXB and AYB such that $AXB\cdot H=0$ and $AYB\cdot K=0$. Let X_1 and Y_1 denote the first points of AXB and AYB, respectively, which belong to the boundary of S_2. Thus X_1 and Y_1 belong to $K-T$ and $H-T$, respectively, and hence are not contiguous to each other. Let AX_1 and AY_1 denote intervals of AXB and AYB, respectively. The set AX_1+AY_1 contains an arc $X_1A_1Y_1$ such that $X_1A_1\cdot H=0$ and $A_1Y_1\cdot K=0$. If X_1 is contiguous to any point of S_2, let X_2 denote such a point, and let X_1X_2 denote the arc consisting of these two points. Otherwise, let R denote a connected domain containing X_1 but containing no point or boundary point (other than X_1) of $H+A_1Y_1$, let X_2 denote a point of $R\cdot S_2$, and let X_1X_2 denote an arc lying in R. If Y_1 is contiguous to any point of S_2, let Y_2 denote such a point, and let Y_1Y_2 denote the arc consisting of these two points. Otherwise, let W denote a connected domain containing Y_1 but containing no point or boundary point (other than Y_1) of $K+A_1X_1+X_1X_2$, let Y_2 denote a point of $W\cdot S_2$, and let Y_1Y_2 denote an arc lying in W. Let X_2Y_2 denote either the point X_2 or an arc lying in S_2 according as X_2 is or is not Y_2. It may be readily shown that the set $X_1A_1Y_1+X_1X_2+Y_1Y_2+X_2Y_2$ contains a simple closed curve J such that (1) J contains A_1 and a point B_1 of X_2Y_2, and (2) of the two segments of J from A_1 to B_1, one contains no point of H and the other contains no point of K. Let I denote that complementary domain of J which does not contain T, and let $H_1=H\cdot(J+I)$ and $K_1=K\cdot(J+I)$. It follows from Theorem 27 that there exists an arc A_1B_1 lying in $J+I-(H_1+K_1)$. Thus A_1B_1 contains no point of M. But this is impossible since A_1 and B_1 lie in different complementary domains of M. The theorem is thus established.

THEOREM 29. *If no point of the arc XY separates the point A from the point B, then XY does not separate A from B.*

Proof. Suppose $S-XY=S_A+S_B$, where S_A and S_B are mutually separated sets containing A and B, respectively. Let S_1 denote the set consisting of X together with all points Z, if there are any, such that the interval XZ does not separate A from B, and let $S_2=XY-S_1$. Now S_2 contains Y, and clearly every point of S_1 precedes every point of S_2. Hence there exists either a last point of S_1 or a first point of S_2.

Suppose there exist a point O which is the last point of S_1 and Q which is the first point of S_2. By Theorem 20 the interval OQ does not separate A from B. Hence by Theorem 28, S_1+OQ does not separate A from B. Thus XQ does not separate A from B, contrary to the fact that Q is a point of S_2.

Suppose that there exists a point O which is the last point of S_1. Let O_1, O_2, O_3, $\cdots$ be a sequence of points of S_2 converging to O such that each precedes the next in the order Y to X. By Theorem 28, the interval OO_n separates A from B. Thus, by Theorem 5, Chapter 2, of P.S.T., the point O separates A from B.

Suppose there exists a point O which is the first point of S_2. Let O_1, O_2, O_3, $\cdots$ be a sequence of points of S_1 converging to O such that each precedes the next in the order X to Y. By Theorem 28, the interval O_nO separates A from B. Hence by Theorem 5, Chapter 2, of P.S.T., the point O separates A from B. Again a contradiction is reached and the theorem is established.

The following proposition is false:

PROPOSITION. *If A, B, and C are three distinct points, and if A is not contiguous to either B or C, then there exists a simple closed curve or triune which separates A from $B+C$.*

In Example 1, let A be any point of the triune $T_1T_2T_3$, and let B and C be points of different complementary domains of this triune. There does not exist a simple closed curve or triune which separates A from $B+C$.

AGRICULTURAL AND MECHANICAL COLLEGE OF TEXAS,
COLLEGE STATION, TEXAS